14.7 Extreme values and Saddle points.

Def. (local maximum)

$f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,y)$ for all (x,y) in an open disk centered at (a,b).

Theorem (1st derivative test for local extreme values)

If $f(x,y)$ has a local maximum or minimum at an interior pt (a,b) of its domain and if the first partial exist, then

$$f_x(a,b) = f_y(a,b) = 0.$$

Remark: consider the tangent plence of the surface $f(x,y) = z$ at $(a,b,f(a,b))$, we will have the equation of this tangent plane

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0$$

i.e. $z = f(a,b)$, which means this plane is horizontal.

Def. (Critical pt)

An interior pt of the domain of $f(x,y)$ is a critical point if (i) $f_x = f_y = 0$ or (ii) one or both of f_x, f_y do not exist.

(Saddle pt)

A differentiable function $f(x,y)$ has a saddle pt at a critical pt if in every open disk centered at (a,b), there exist (x_1,y_1) s.t. $f(x_1,y_1) > f(a,b)$ and (x_2,y_2) s.t. $f(x_2,y_2) < f(a,b)$.

Remark: (i) If f_x and f_y exist, then

$$\text{local extremes} \begin{cases} f_x = f_y = 0 \\ \text{boundary values} \end{cases}$$

(ii)

$$\text{critical pts} \begin{cases} f_x = f_y = 0 \begin{cases} \text{local extremes} \\ \text{saddle point.} \end{cases} \\ \text{one or both of } f_x, f_y \text{ do not exist.} \end{cases}$$

Thm (2nd derivative test of local extremes)

Suppose $f(x,y)$, f_x, f_y continuous in a disk centered at (a,b), and $f_x(a,b)=f_y(a,b)=0$,

then (i) f has a local maximum at (a,b) if

$f_{xx}<0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ at (a,b).

(ii) " minimum "

$f_{xx}>0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ at (a,b)

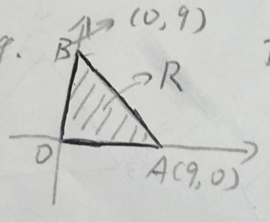
(iii) saddle point if $f_{xx}f_{yy}-f_{xy}^2<0$ at (a,b).

(iv) inconclusive if $f_{xx}f_{yy}-f_{xy}^2=0$ at (a,b).

Steps to find absolute maxima and minima on closed bounded regions.

1. List the interior pts of R where f may have local extremes.
2. List the boundary pts of R where f has local extremes (with respect to the boundary of R)
3. Compare all of them.

Remark: e.g.



$f(x,y)=2+2x+2y-x^2-y^2$

Step 1. $f_x=f_y=0 \Rightarrow (1,1)$, $f(1,1)=4$

Step 2. (i) On OA, $f(x,y)=f(x,0)=2+2x-x^2$.

local extremes: $(0,0)$, $f(0,0)=2$

$(9,0)$, $f(9,0)=-61$

(ii) OB

(iii) AB

Step 3. ________.

Example: Find the maximum value of $s = xy + yz + zx$ where $x + y + z = 6$.

Solution: $s = xy + (x+y)(6-x-y) = 6(x+y) - xy - x^2 - y^2$, $(x,y) \in \mathbb{R}^2$.

Let $f(x,y) = s = 6(x+y) - xy - x^2 - y^2$.

Since $\lim_{x \to \infty} f(x,y) = \lim_{y \to \infty} f(x,y) = -\infty$, then the maximum of s must be the local maximum.

$$\begin{cases} f_x = 6 - y - 2x = 0 \\ f_y = 6 - x - 2y = 0 \end{cases}, \quad \text{then } (x,y) = (2,2).$$

$f_{xx} = -2 < 0 \quad f_{yy} = -2 \quad f_{xy} = -1$

$f_{xx} f_{yy} - f_{xy}^2 = 3 > 0.$

Hence, $(2,2)$ is the local maximum of f.

Thus, $(2,2)$ is the maximum of f.

$S_{max} = f(2,2) = 12.$